1 Marion and Thornton Chapter 7

Hamilton’s Principle - Lagrangian and Hamiltonian dynamics.

1.1 Problem 7.37

Fig. 1 shows a double Atwood machine. What is the tension in the strings (here of length $a$ and $b$)?

In Example 7.8 of Marion and Thornton Chapter 7 an Atwood machine is done using the Lagrangian method. It doesn’t use Lagrange multipliers. All that is being asked now is that we repeat the problem using Lagrangian multipliers. We keep $x_2$ and $y_2$ as seen in Fig. 1 as variables and use the constraints on rope $a$ and rope $b$: $f_a(x_1, x_2) = x_1 + x_2 - a = 0 \quad \text{(1a)}$

$f_b(y_1, y_2) = y_1 + y_2 - b = 0. \quad \text{(1b)}$

The Lagrangian is then

$U = -m_1 g x_1 - m_2 g (x_2 + y_1) - m_3 g (x_2 + y_2) \quad \text{(2a)}$

$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_2 + \dot{y}_1)^2 + \frac{1}{2} m_3 (\dot{x}_2 + \dot{y}_2)^2 \quad \text{(2b)}$

$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_2 + \dot{y}_1)^2 + \frac{1}{2} m_3 (\dot{x}_2 + \dot{y}_2)^2$

$+ m_1 g x_1 + m_2 g (x_2 + y_1) + m_3 g (x_2 + y_2) \quad \text{(2c)}$

Now then the Lagrange equations of motion are
Lagrange Equation for $y_1$

$$0 = \partial L / \partial y_1 - \frac{d}{dt} \partial L / \partial \dot{y}_1 + \lambda_a \partial f_a / \partial y_1 + \lambda_b \partial f_b / \partial y_1$$

$$= m_2 g - \frac{d}{dt} [m_2 (\ddot{x}_2 + \ddot{y}_1)] + 0 + \lambda_b$$

$$\ddot{x}_2 + \ddot{y}_1 = g + \frac{\lambda_b}{m_2} \quad (3a)$$

Lagrange Equation for $y_2$

$$0 = \partial L / \partial y_2 - \frac{d}{dt} \partial L / \partial \dot{y}_2 + \lambda_a \partial f_a / \partial y_2 + \lambda_b \partial f_b / \partial y_2$$

$$= m_2 g - \frac{d}{dt} [m_3 (\ddot{x}_2 + \ddot{y}_2)] + 0 + \lambda_b$$

$$\ddot{x}_2 + \ddot{y}_2 = g + \frac{\lambda_b}{m_3} \quad (3b)$$

Lagrange Equation for $x_1$

$$0 = \partial L / \partial x_1 - \frac{d}{dt} \partial L / \partial \dot{x}_1 + \lambda_a \partial f_a / \partial x_1 + \lambda_b \partial f_b / \partial x_1$$

$$= m_1 g - \frac{d}{dt} [m_1 \ddot{x}_1] + \lambda_a + 0$$

$$= m_1 g - m_1 \ddot{x}_1 + \lambda_a$$

$$\ddot{x}_1 = g + \frac{\lambda_a}{m_1} \quad (3c)$$

Lagrange Equation for $x_2$

$$0 = \partial L / \partial x_2 - \frac{d}{dt} \partial L / \partial \dot{x}_2 + \lambda_a \partial f_a / \partial x_2 + \lambda_b \partial f_b / \partial x_2$$

$$= m_2 g + m_3 g - \frac{d}{dt} [m_2 (\ddot{x}_2 + \ddot{y}_1) + m_3 (\ddot{x}_2 + \ddot{y}_2)] + \lambda_a$$

$$\lambda_a = m_2 (\ddot{x}_2 + \ddot{y}_1) + m_3 (\ddot{x}_2 + \ddot{y}_2) - (m_2 + m_3) g \quad (3d)$$

One last thing, notice from the constraints Eq. (1) that

$$\ddot{x}_1 + \ddot{x}_2 = 0 \quad (4a)$$

$$\ddot{y}_1 + \ddot{y}_2 = 0 \quad (4b)$$
which means that we can rewrite Eq. (3) completely in terms of \( x_1 \) and \( y_1 \)

\[
\ddot{x}_1 - \ddot{y}_1 = - \left( g + \frac{\lambda_b}{m_2} \right) \tag{5a}
\]

\[
\ddot{x}_1 + \ddot{y}_1 = - \left( g + \frac{\lambda_b}{m_3} \right) \tag{5b}
\]

\[
0 = g + \frac{\lambda_a}{m_1} - \ddot{x}_1 \tag{5c}
\]

\[
\lambda_a = - m_2 (\ddot{x}_1 - \ddot{y}_1) - m_3 (\ddot{x}_1 + \ddot{y}_1) - (m_2 + m_3) g. \tag{5d}
\]

Substituting the first two (Eq. (5a) and Eq. (5b)) into the last gives

\[
\lambda_a = - m_2 (\ddot{x}_1 - \ddot{y}_1) - m_3 (\ddot{x}_1 + \ddot{y}_1) - (m_2 + m_3) g
\]

Then substitute \( \lambda_a \) into Eq. (5c) to get

\[
0 = g + \frac{\lambda_a}{m_1} - \ddot{x}_1
\]

\[
= g + \frac{\lambda_a}{m_1} + g + \frac{\lambda_b}{2m_2} + \frac{\lambda_b}{2m_3}
\]

\[
= 2g + \frac{2\lambda_b}{m_1} + \frac{\lambda_b}{2m_2} + \frac{\lambda_b}{2m_3}
\]

\[
= 4g + \lambda_b \left( \frac{4}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right)
\]

\[
\lambda_b = - \frac{4g}{\frac{4}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}}
\]

\[
\lambda_b = - \frac{4g m_1 m_2 m_3}{4m_2 m_3 + m_1 m_3 + m_1 m_2}. \tag{6}
\]

We’ve done all the work now we just identify the forces that ensure the constraints are met are the tensions in the ropes \( a \) and \( b \). The tension in the first rope \( a \) on \( m_1 \) is

\[
F_a = \lambda_a \frac{\partial f_a}{\partial x_1} = \lambda_a = 2\lambda_b \tag{6a}
\]

\[
F_a = - \frac{8g m_1 m_2 m_3}{4m_2 m_3 + m_1 m_3 + m_1 m_2}. \tag{8a}
\]
The tension on robe $b$ for mass $m_2$ is

$$F_b = \lambda_b \frac{\partial f_b}{\partial y_1} = \lambda_b$$

$$F_b = -\frac{4gm_1m_2m_3}{4m_2m_3 + m_1m_3 + m_1m_2}$$

and of course is the same for mass $m_3$.

Note the solutions for $\lambda_a$ and $\lambda_b$ mean that

$$\ddot{x}_1 = g + \frac{\lambda_a}{m_1}$$

$$\ddot{x}_1 = g \left(1 - \frac{4m_2m_3}{4m_2m_3 + m_1m_3 + m_1m_2}\right)$$

(9a)

$$\ddot{x}_2 = -\ddot{x}_1$$

(9b)

$$\ddot{y}_1 = -\left(g + \frac{\lambda_b}{m_3}\right) - \ddot{x}_1$$

$$\ddot{y}_1 = - \left(g + \frac{\lambda_b}{m_3}\right) - g - \frac{2\lambda_b}{m_1}$$

$$\ddot{y}_1 = -2g - \frac{m_1 + 2m_3}{m_1 m_3} \lambda_b$$

$$\ddot{y}_1 = -2g + \frac{m_1 + 2m_3}{m_1 m_3} \frac{4gm_1m_2m_3}{4m_2m_3 + m_1m_3 + m_1m_2}$$

$$\ddot{y}_1 = 2g \left(\frac{2m_1m_2 + 4m_2m_3}{4m_2m_3 + m_1m_3 + m_1m_2} - 1\right)$$

(9c)

$$\ddot{y}_2 = -\ddot{y}_1$$

(9d)